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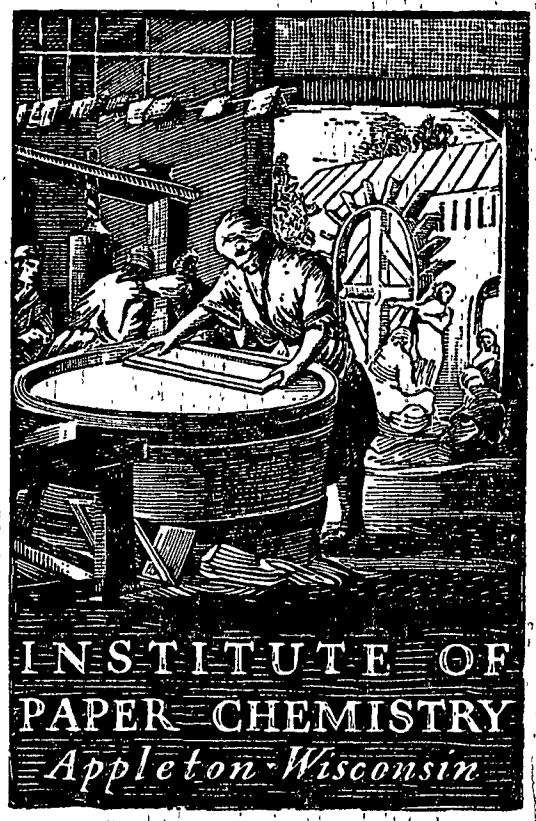

STUDIES OF THE SHEET-FORMING PROCESS

Approximate Theories of the Filtration Process

Project 2348

Report Five

A Progress Report

to

MEMBERS OF GROUP PROJECT 2348

March 23, 1964

THE INSTITUTE OF PAPER CHEMISTRY
Appleton, Wisconsin

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THE INSTITUTE OF PAPER CHEMISTRY
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STUDIES OF THE SHEET-FORMING PROCESS

APPROXIMATE THEORIES OF THE FILTRATION PROCESS

SUMMARY

A scheme is proposed for the macroscopic description of fluid flow through deformable porous solids, in which it has been attempted to introduce the assumptions pertaining to special cases in a logical sequence, so that it will not be difficult to retrace the argument to the appropriate point when modifications are required. It has been assumed, throughout, that accumulation in the porous solid may take place during flow. The discussion is not restricted, in any essential way, to the flow of fluids through porous solids consisting of fibers, although this was the application of most immediate interest.

The distinguishing features of the present treatment are: (a) the proper accounting for resistance to flow when the solid phase is in motion, (b) a suggested method by which the most significant of the possible inertial effects may be included, and (c) the description of the system, in a specific case, by means of a boundary value problem involving a single partial differential equation, in a form amenable to numerical solution.

By an extension of the methods developed for the flow problem, a theory is developed for the retention of small particles in a deformable porous solid, under increasingly specific assumptions. An explicit solution is derived in a simple but important special case, in which the flow is steady or nearly steady and there is, at most, slow accumulation in the porous solid.

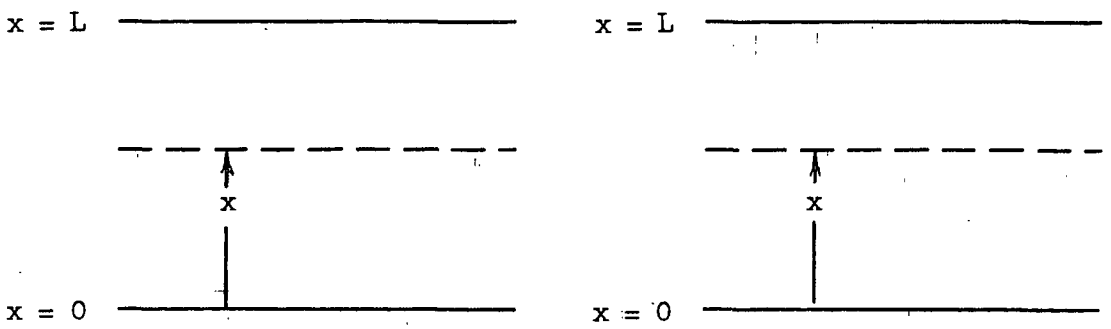
PART I. FLOW OF FLUIDS THROUGH DEFORMABLE POROUS SOLIDS

DESCRIPTION OF THE MECHANICAL SYSTEM

We will give, at first, a more general statement of the conditions of the problem than the intended applications require. The system to be considered consists of a porous solid through which a fluid is made to flow. It will be described, as is customary and virtually necessary, in terms of variables which are local averages; the way in which these correspond to the microscopic variables will usually be obvious. Moreover, it will be supposed that a rectangular co-ordinate frame can be chosen, such that the quantities in question depend only on one spatial co-ordinate and on the time. Clearly the fluid flow, apart from small-scale irregularities, must take place in this preferred direction.

For definiteness, such a rectangular co-ordinate frame will be chosen and the independent variables taken to be x and t. It will become evident that the general conclusions stated herein (with an exception which will be noted) do not require any special choice of the origin of the x axis; with respect to the laboratory frame it may be a moving (and even an accelerated) point. In some applications this flexibility is very convenient.

As an aid in defining notation, we may consider a case in which the system extends from $x = 0$ to $x = L$. It is helpful to consider the two components of the system separately, and we will represent them as physically separated (Fig. 1). The quantities associated with the fluid phase (identified, where necessary, by the subscripts λ or Λ) appear at the left, while those associated with the solid phase (the identifying subscripts being ϕ or Φ) appear at the right. In any small portion of the system, the fluid is to be thought of as filling the fraction $\varepsilon(x,t)$ of the available space, since it is excluded from a large number



FLUID

fluid fraction

$$\varepsilon(x,t)$$

true density

$$\rho_{\Lambda}$$

apparent density

$$\rho_{\lambda}(x,t) = \rho_{\Lambda}\,\varepsilon(x,t)$$

pressure

$$p_{\lambda}(x,t)$$

internal superficial velocity

$$U_{\lambda}(x,t) = \varepsilon(x,t)\, v_{\lambda}(x,t)$$

SOLID

solid fraction

$$1 - \varepsilon(x,t)$$

true density

$$\rho_{\Phi}$$

apparent density

$$\rho_{\phi}(x,t) = \rho_{\Phi}\,[1 - \varepsilon(x,t)]$$

pressure

$$p_{\phi}(x,t)$$

internal superficial velocity

$$U_{\phi}(x,t) = [1 - \varepsilon(x,t)]\, v_{\phi}(x,t)$$

Figure 1. Components of the System, and Their Variables

of small interior regions which are, of course, identical with the regions occupied by the solid.

A distinction has been made between the true density of the fluid (mass of fluid per unit volume of fluid) and its apparent density (mass of fluid per unit volume of the system). The internal superficial velocity of the fluid is defined as the volume of fluid which flows, per unit time, across a plane surface normal to the x axis, per unit area of the surface. It has the dimensions of a velocity, and is given a positive or negative sign according as the flow takes place in the positive or negative direction of the x axis. In this connection we also define a quantity v_λ, which is an averaged microscopic fluid velocity (see Fig. 1). The fluid fraction (or porosity), the apparent density, the pressure, and the internal superficial velocity are local averages, taken over small volumes, which are, however, of such extent that fluctuations in the resulting quantities are not appreciable. It is essential to the adequacy of the model that the distances typical of such volumes should not greatly exceed the distances within which the resulting averages undergo significant change.

Similar remarks apply to the quantities shown in connection with the solid phase. While it is obviously a considerable simplification, the state of stress in the solid phase will be described by means of a single quantity, the pressure p_ϕ. To complete the description, it is necessary to specify the ways in which the two phases may influence each other, for example through the drag forces resulting from relative motion.

In a special case which will be discussed in detail, the system is considered to be confined between $x = 0$ and $x = L$ by permeable boundaries (in which we include the possibility that the boundary at $x = L$ may consist physically of nothing more than discontinuities in some of the quantities which describe the

system.) As an arbitrary convention, it will be supposed that the prevailing fluid flow is in the direction of the negative x axis; thus, the various velocities mentioned above will ordinarily be negative. The fluid approaching the system may carry in suspension particles identical with those which make up the solid phase of the system; for this reason the thickness L and the total mass per unit area M may change with time. Equations which pertain to the special case here described will appear with the letter (a) following the equation number; they differ from equations not so designated only in that the material of the solid phase is assumed to be stationary at $x = 0$.

The boundary conditions for the problem may consist of some combination of the following quantities (which will be given names suggestive of experimental practice): the upstream fluid pressure, $p_\lambda(L,t)$; the downstream fluid pressure, $p_\lambda(0,t)$; the pressure exerted (for example, by a grid) on the upper boundary of the solid phase, $p_\phi(L,t)$; the pressure exerted on the lower boundary of the solid phase, $p_\phi(0,t)$; the fluid inflow rate per unit area, $-U_\lambda(L,t)$; the fluid outflow rate per unit area, $-U_\lambda(0,t)$. Unless the problem involves a steady state, initial conditions will also be required, which may take the form of specified values of some combination of the variables listed in Fig. 1, at a given value of the time.

THE CONTINUITY CONDITIONS

In the usual way, continuity conditions may be derived for each of the phases. These will be stated in terms of the averaged variables which have been chosen to describe the system. Assuming that the fluid may be compressible (i.e. that ρ_Λ is not necessarily constant), we obtain

$$\frac{\partial}{\partial x}\left[\rho_\Lambda U_\lambda\right] + \frac{\partial}{\partial t}\left[\rho_\Lambda \varepsilon\right] = 0 \qquad (1).$$

Likewise, assuming that the solid phase may be compressible (in the sense that $\rho_{\bar{\phi}}$ is not necessarily constant), we obtain

$$\frac{\partial}{\partial x}\left[\rho_{\bar{\phi}}\, U_{\phi}\right] + \frac{\partial}{\partial t}\left[\rho_{\bar{\phi}}\,(1 - \varepsilon)\right] = 0 \qquad (2).$$

When the fluid is incompressible, Equation (1) becomes

$$\frac{\partial U_{\lambda}}{\partial x} = -\frac{\partial \varepsilon}{\partial t} \qquad (3).$$

When the solid phase is incompressible, Equation (2) becomes

$$\frac{\partial U_{\phi}}{\partial x} = \frac{\partial \varepsilon}{\partial t} \qquad (4).$$

If both fluid and solid phases are incompressible, we have

$$\frac{\partial U_{\lambda}}{\partial x} = -\frac{\partial U_{\phi}}{\partial x} \qquad (5),$$

which, on integration with respect to x, gives

$$U_{\lambda}(x,t) = U_{\lambda}(0,t) - \left[U_{\phi}(x,t) - U_{\phi}(0,t)\right] \qquad (6).$$

In the special case described above, the solid phase has an internal superficial velocity of zero at $x = 0$, and Equation (6) becomes

$$U_{\lambda}(x,t) = U_{\lambda}(0,t) - U_{\phi}(x,t) \qquad (7).(a).$$

The usefulness of Equations (6) and (7) corresponds to the practical importance of problems in which both phases are substantially incompressible, and $U_{\lambda}(0,t)$ is one of the prescribed conditions. It should be noted that it is <u>not</u> being assumed, in this context, that the solid fraction $1 - \varepsilon(x,t)$ is constant; the solid phase

may be said to be "compressible" in the sense that its apparent density may vary, but "incompressible" in the sense that its true density does not. To avoid this ambiguity, the solid phase will be said to be deformable.

PRESSURE RELATIONSHIPS

If the force required to produce acceleration in the x direction, in either component, can be ignored, a simple relation exists connecting the two pressure variables. We find

$$\frac{\partial p_\phi}{\partial x} = -\frac{\partial p_\lambda}{\partial x} \qquad (8),$$

so that

$$p_\phi(x,t) - p_\phi(0,t) = -[p_\lambda(x,t) - p_\lambda(0,t)] \qquad (9),$$

and

$$p_\phi(x,t) - p_\phi(L,t) = -[p_\lambda(x,t) - p_\lambda(L,t)] \qquad (10).$$

For greater generality, let us now suppose that the force required to accelerate the fluid in the x direction, per unit volume of the system, is F_λ. If the force, per unit volume of the system, resulting from relative motion of the two phases, is denoted by $-\Pi(x,t)$, this being the force exerted on the solid phase by the fluid phase, we have

$$F_\lambda = -\frac{\partial p_\lambda}{\partial x} + \Pi \qquad (11).$$

Similarly, let the force required to accelerate the solid phase in the x direction, per unit volume of the system, be F_ϕ. Then

$$F_{\phi} = -\frac{\partial p_{\phi}}{\partial x} - \Pi \qquad (12).$$

In the remainder of this section, an inertial co-ordinate frame will be assumed. It is suggested that the following may be useful approximations for the forces in question:

$$F_{\lambda} \approx \rho_{\lambda} \left[\frac{\partial v_{\lambda}}{\partial t} + v_{\lambda} \frac{\partial v_{\lambda}}{\partial x}\right] \qquad (13),$$

and

$$F_{\phi} \approx \rho_{\phi} \left[\frac{\partial v_{\phi}}{\partial t} + v_{\phi} \frac{\partial v_{\phi}}{\partial x}\right] \qquad (14).$$

Here, Equation (14) probably represents a closer estimate; obviously, neither expression is exact. Equation (15) should become correct in the limit that the solid fraction is small, and it is believed that Equations (13) and (14) will ordinarily correspond to small effects; of these, the latter will be negligible in many cases.

FLOW RESISTANCE

When the material of the solid phase is not in motion, and the effects of acceleration discussed in the previous section are neglected, it is known that the internal superficial velocity of the fluid (if sufficiently small in magnitude) is proportional to the fluid pressure gradient:

$$U_{\lambda} = -\frac{1}{\alpha} \frac{\partial p_{\lambda}}{\partial x} \qquad (15),$$

where $\alpha = \mu r \rho_{\phi}$, in which μ is the fluid viscosity and r is the local flow resistance. Let it be assumed, for the present, that this relation is exact.

The quantity which is to be related to the fluid pressure gradient when the solid phase is in motion must be, in effect, the velocity difference

$$v_\lambda - v_\phi = \frac{U_\lambda}{\varepsilon} - \frac{U_\phi}{1 - \varepsilon} \qquad (16).$$

To secure agreement with the customary definition of r, consideration of the appropriate limiting case shows that the correct generalization of Equation (15) is

$$U_\sigma = U_\lambda - \frac{\varepsilon}{1 - \varepsilon} U_\phi = - \frac{1}{\alpha} \frac{\partial p_\lambda}{\partial x} \qquad (17).$$

The quantity on the left will be called the relative superficial velocity. Finally, to take into account the effects of acceleration discussed above, we revise Equation (17) to read

$$U_\sigma = U_\lambda - \frac{\varepsilon}{1 - \varepsilon} U_\phi = - \frac{1}{\alpha} \Pi \qquad (18).$$

Returning to the situation to which Equation (15) applies, it is found that higher internal superficial velocities can be accommodated if a quadratic term is added:

$$- \frac{\partial p_\lambda}{\partial x} = \alpha U_\lambda + \beta U_\lambda^2 \, \mathrm{sgn}(U_\lambda) \qquad (19).$$

Here, α and β are semiempirical functions of ε and other system parameters (an example of such a correlation will be given later), and $\mathrm{sgn}(z)$ has the value $+1$ if $z \geq 0$ and the value -1 if $z < 0$. Proceeding as above, we obtain

$$- \frac{\partial p_\lambda}{\partial x} = \alpha U_\sigma + \beta U_\sigma^2 \, \mathrm{sgn}(U_\sigma) \qquad (20),$$

corresponding to Equation (17), and

$$-\Pi = \alpha U_\sigma + \beta U_\sigma^2 \operatorname{sgn}(U_\sigma) \qquad (21),$$

corresponding to Equation (18).

It is customary to refer to the first term on the right in Equation (19) as a "viscous" term, and to the second as an "inertial" term. The inertial effects which find expression in the latter are to be distinguished from those discussed in the previous section, and, in many cases, may require inclusion even when those discussed previously are negligible.

EXAMPLE

To illustrate the meaning of the various velocities which have been introduced, let the solid phase have the form of a homogeneous plug, which is made to slide, with a velocity w_2, through a tube.

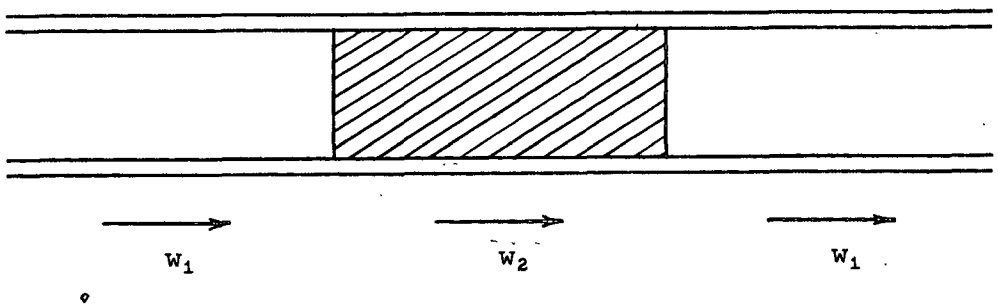


Figure 2. Flow in Nondeformable Porous Solid

Outside the plug, the tube is filled with liquid, which is moving with a velocity w_1; the liquid also fills the pores of the solid. It is assumed that the plug moves as a rigid body.

At points outside the plug, it is clear that $\varepsilon = 1$, $v_\phi = 0$, $U_\phi = 0$, $v_\lambda = w_1$, and $U_\lambda = w_1$.

At points within the plug, we have $v_\phi = w_2$. Thus, if the porosity of the plug is ε, the internal superficial velocity of the solid phase is $U_\phi = (1 - \varepsilon)w_2$. It is not difficult to see that $v_\lambda - v_\phi = (w_1 - w_2)/\varepsilon$, from which $v_\lambda = w_1/\varepsilon - (1 - \varepsilon)w_2/\varepsilon$. Thus, the internal superficial velocity of the fluid is found to be $U_\lambda = w_1 - (1 - \varepsilon)w_2$, and the relative superficial velocity is $U_\sigma = w_1 - w_2$.

It may be instructive to compare two instances of the foregoing. In experiment (A) the plug is stationary and the liquid moves at a velocity w at points outside the plug; in experiment (B) the liquid outside the plug is stationary and the plug moves with a velocity $-w$. We list the values of the various velocities at points inside the plug.

(A)	(B)
$w_1 = w$	$w_1 = 0$
$w_2 = 0$	$w_2 = -w$
$v_\phi = 0$	$v_\phi = -w$
$U_\phi = 0$	$U_\phi = -(1 - \varepsilon)w$
$v_\lambda = w/\varepsilon$	$v_\lambda = (1 - \varepsilon)\, w/\varepsilon$
$U_\lambda = w$	$U_\lambda = (1 - \varepsilon)w$
$U_\sigma = w$	$U_\sigma = w$

APPLICATION TO FLOW THROUGH FIBER MATS

In the section which follows the general considerations presented above will be applied to the flow of liquids through fiber mats. Several simplifying assumptions will be made, which include the omission of both kinds of inertial effects. Each phase will be considered incompressible, permitting use of Equations

(3) through (7). The relation between the two pressure variables will be as shown in Equations (8) through (10).

Recent experimental work (1) has indicated that the following forms for the coefficients in Equation (20) are appropriate:

$$\alpha = a\mu\rho_{\bar{\Phi}}\mathcal{V}S_V^2(1-\varepsilon)^{3/2}[1+b(1-\varepsilon)^3] \qquad (22),$$

and

$$\beta = b'a^{1/2}\rho_{\Lambda}\,\rho_{\bar{\Phi}}\mathcal{V}S_V(1-\varepsilon)^{3/4}\,\varepsilon^{-3/2}[1+b(1-\varepsilon)^3]^{1/2} \qquad (23).$$

The simplification chosen is to set β equal to zero, i.e. the relation shown in Equation (17) is assumed.

Information is also required, as to the dependence of the solid fraction on the other variables. The relation which will be assumed,

$$\rho_{\bar{\Phi}}(1-\varepsilon) = c_o + c_1 p_{\phi}^{N} \qquad (24),$$

is taken from studies of the effects of static loads on fiber mats. In the absence of evidence to the contrary, it will be considered that this expression (in which the time does not appear explicitly) may be adequate even for the description of rapidly varying conditions.

In the present section, the rectangular co-ordinate frame will be used, the independent variables being x and t. It will be supposed that the solid phase is stationary at $x = 0$, as assumed in obtaining Equation (7). As described earlier, the thickness L and the total mass per unit area M may depend on the time. At the risk of emphasizing the obvious, we assume that the apparent density of the solid phase, which appears at the left in Equation (24), will never be less than c_o at points within the system, which extends from $x = 0$ to $x = L$.

With the aid of Equation (7), we have

$$U_\sigma = U_\lambda - \frac{\varepsilon}{1-\varepsilon} U_\phi = U_\lambda(x,t) - \frac{\varepsilon}{1-\varepsilon}[U_\lambda(0,t) - U_\lambda(x,t)] \qquad (25)\ (a),$$

or

$$U_\sigma = \frac{1}{1-\varepsilon} U_\lambda(x,t) - \frac{\varepsilon}{1-\varepsilon} U_\lambda(0,t) \qquad (26)\ (a).$$

After rearrangement, Equation (17) becomes

$$U_\lambda(x,t) = \varepsilon U_\lambda(0,t) - \frac{1-\varepsilon}{\alpha} \frac{\partial p_\lambda}{\partial x} \qquad (27)\ (a).$$

Substituting this result in Equation (3), we obtain

$$\frac{\partial \varepsilon}{\partial t} = -\frac{\partial}{\partial x}\left[\varepsilon U_\lambda(0,t) - \frac{1-\varepsilon}{\alpha} \frac{\partial p_\lambda}{\partial x}\right] \qquad (28)\ (a),$$

or

$$\frac{\partial \varepsilon}{\partial t} = -U_\lambda(0,t) \frac{\partial \varepsilon}{\partial x} - \frac{\partial}{\partial x}\left[\frac{1-\varepsilon}{\alpha} \frac{\partial p_\phi}{\partial x}\right] \qquad (29)\ (a),$$

in which we have applied Equation (8).

Equation (29) may be considered as a partial differential equation in the dependent variable ε ; the necessary derivatives, which are somewhat complicated, can be deduced from Equations (22) and (24).

There are advantages, however, in conversion of the result to an equation in which p_ϕ is, in effect, the dependent variable. We first rewrite Equation (29) as

$$-\frac{\partial \rho_\phi}{\partial t} = U_\lambda(0,t)\frac{\partial \rho_\phi}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\rho_\phi}{\alpha}\frac{\partial p_\phi}{\partial x}\right] \qquad (30),(a),$$

in which

$$\rho_\phi = \rho_{\bar{\phi}}(1-\varepsilon) = c_o + c_1 p_\phi^N ,$$

as in Equation (24). For brevity, let

$$\zeta(p_\phi) = \frac{d\rho_\phi}{dp_\phi} \qquad (31)$$

Equation (30) becomes

$$-\zeta\frac{\partial p_\phi}{\partial t} = U_\lambda(0,t)\,\zeta\frac{\partial p_\phi}{\partial x} - \frac{\partial}{\partial x}\left[\frac{\rho_\phi}{\alpha}\frac{\partial p_\phi}{\partial x}\right] \qquad (32),(a).$$

Now let a new dependent variable $\psi(p_\phi)$ be chosen, such that

$$\frac{d\psi}{dp_\phi} = \frac{\rho_\phi}{\alpha}; \qquad \psi(0) = 0 \qquad (33).$$

We then have, from Equation (32),

$$-\zeta\frac{\alpha}{\rho_\phi}\frac{\partial \psi}{\partial t} = \zeta\frac{\alpha}{\rho_\phi}U_\lambda(0,t)\frac{\partial \psi}{\partial x} - \frac{\partial^2\psi}{\partial x^2} \qquad (34),(a),$$

or

$$\frac{\partial \psi}{\partial t} = -U_\lambda(0,t)\frac{\partial \psi}{\partial x} + \frac{\rho_\phi}{\zeta\alpha}\frac{\partial^2\psi}{\partial x^2} \qquad (35),(a),$$

which is now a quasi-linear partial differential equation of second order. The solution of this equation, by numerical methods, will require suitable initial and boundary values; one boundary occurs at $x = 0$ and is fixed in position, but the

other, at $x = L$, may move. When it does not, Equation (35) appears to be the simplest and most easily manageable formulation of the problem. When the outflow rate $-U_\lambda(0,t)$ is constant, further formal simplification is possible, by introduction of a moving co-ordinate system $[x'' = x - U_\lambda(0,t)t\,,\ t'' = t]$, giving

$$\frac{\partial \psi}{\partial t''} = \frac{\rho_\phi}{\zeta \alpha} \frac{\partial^2 \psi}{\partial x''^2} \qquad (36)\ (a)\,.$$

There is a disadvantage in this, that one has two moving boundaries and the attendant complications in numerical procedures.

When L is variable, but is a <u>known</u> function of the time, the problem can be recast so as to yield an equation to be solved with fixed boundaries, in which the independent variables are $x' = x/L$ and $t' = t$. When the substitutions

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{x}{L^2}\frac{dL}{dt}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - \frac{x'}{L}\frac{dL}{dt'}\frac{\partial}{\partial x'} \qquad (37)$$

and

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \qquad (38)$$

are made in Equation (32), we obtain, after rearrangement,

$$-\zeta \frac{\partial p_\phi}{\partial t'} = \left[U_\lambda(0,t')\frac{1}{L} - \frac{x'}{L}\frac{dL}{dt'}\right]\zeta \frac{\partial p_\phi}{\partial x'} - \frac{1}{L}\frac{\partial}{\partial x'}\left[\frac{\rho_\phi}{\alpha L}\frac{\partial p_\phi}{\partial x'}\right] \qquad (39)\ (a)\,.$$

By the same process as before, and with the same definitions of ζ and ψ, the result is

$$\frac{\partial \psi}{\partial t'} = -\left[U_\lambda(0,t')\frac{1}{L} - \frac{x'}{L}\frac{dL}{dt'}\right]\frac{\partial \psi}{\partial x'} + \frac{\rho_\phi}{L^2 \zeta \alpha}\frac{\partial^2 \psi}{\partial x'^2} \qquad (40)\ (a)\,.$$

For convenience, the corresponding conclusions will be derived for a different choice of independent variables; these, which some prefer, are m and τ, defined by

$$m = \int_0^x \rho_{\bar{\phi}} [1 - \varepsilon(x_I,t)]dx_I \quad \text{and} \quad \tau = t \qquad (41).$$

We then have

$$\frac{\partial}{\partial x} = \rho_{\bar{\phi}} [1 - \varepsilon(x,t)] \frac{\partial}{\partial m} = \rho_\phi \frac{\partial}{\partial m} \qquad (42),$$

and

$$\frac{\partial}{\partial t} = \frac{\partial m}{\partial t} \frac{\partial}{\partial m} + \frac{\partial}{\partial \tau} \qquad (43).$$

The latter expression may be arranged in equivalent form with the aid of Equation (3):

$$\frac{\partial m}{\partial t} = -\rho_{\bar{\phi}} \int_0^x \frac{\partial \varepsilon}{\partial t} dx_I = \rho_{\bar{\phi}} \int_0^x \frac{\partial U_\lambda}{\partial x} dx_I \qquad (44),$$

so that

$$\frac{\partial m}{\partial t} = \rho_{\bar{\phi}} [U_\lambda(x,t) - U_\lambda(0,t)] \qquad (45).$$

Equation (30) becomes

$$-\frac{\partial \rho_\phi}{\partial \tau} = [U_\lambda(0,t)\rho_\phi + \frac{\partial m}{\partial t}] \frac{\partial \rho_\phi}{\partial m} - \rho_\phi \frac{\partial}{\partial m} [\frac{\rho_\phi^2}{\alpha} \frac{\partial p_\phi}{\partial m}] \qquad (46)\ (a),$$

and with

$$U_\lambda(x,t) = \varepsilon U_\lambda(0,t) + \frac{1-\varepsilon}{\alpha}\frac{\partial p_\phi}{\partial x} = \varepsilon U_\lambda(0,t) + \frac{1-\varepsilon}{\alpha}\rho_\phi\frac{\partial p_\phi}{\partial m} \qquad (47)\ (a)$$

from Equation (27), together with

$$\zeta(p_\phi) = \frac{d\rho_\phi}{dp_\phi} \qquad (31),$$

as before, we have

$$-\zeta\frac{\partial p_\phi}{\partial \tau} = \frac{\rho_\phi^2}{\alpha}\frac{\partial p_\phi}{\partial m}\zeta\frac{\partial p_\phi}{\partial m} - \rho_\phi\frac{\partial}{\partial m}\left[\frac{\rho_\phi^2}{\alpha}\frac{\partial p_\phi}{\partial m}\right] \qquad (48)\ (a).$$

Let the dependent variable in this case be $\Psi(p_\phi)$, such that

$$\frac{d\Psi}{dp_\phi} = \frac{\rho_\phi^2}{\alpha}; \quad \Psi(0) = 0 \qquad (49).$$

Equation (48) becomes

$$-\zeta\frac{\alpha}{\rho_\phi^2}\frac{\partial\Psi}{\partial\tau} = \frac{\partial\Psi}{\partial m}\zeta\frac{\alpha}{\rho_\phi^2}\frac{\partial\Psi}{\partial m} - \rho_\phi\frac{\partial^2\Psi}{\partial m^2} \qquad (50)\ (a),$$

or

$$\frac{\partial\Psi}{\partial\tau} = -\left[\frac{\partial\Psi}{\partial m}\right]^2 + \frac{\rho_\phi^3}{\zeta\alpha}\frac{\partial^2\Psi}{\partial m^2} \qquad (51)\ (a).$$

The practical aspect of the solution of this nonlinear equation, with appropriate initial and boundary conditions, is similar to that of Equation (35). The boundary at $m = 0$ is fixed, but the other boundary, at $m = M$, may depend on the time. Here M is defined by

$$M = \int_0^L \rho_{\underline{\phi}} [1 - \varepsilon(x_I, t)] dx_I \qquad (52).$$

If M varies, but is a <u>known</u> function of the time, we may introduce new independent variables $m' = m/M$ and $\tau' = \tau = t$ and obtain an equation to be solved with fixed boundaries. The substitutions are

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau'} - \frac{m}{M^2} \frac{dM}{d\tau} \frac{\partial}{\partial m'} = \frac{\partial}{\partial \tau'} - \frac{m'}{M} \frac{dM}{d\tau'} \frac{\partial}{\partial m'} \qquad (53),$$

and

$$\frac{\partial}{\partial m} = \frac{1}{M} \frac{\partial}{\partial m'} \qquad (54).$$

With rearrangement, Equation (48) becomes

$$-\zeta \frac{\partial p_\phi}{\partial \tau'} = [\frac{\rho_\phi^2}{\alpha M^2} \frac{\partial p_\phi}{\partial m'} - \frac{m'}{M} \frac{dM}{d\tau'}] \zeta \frac{\partial p_\phi}{\partial m'} - \frac{\rho_\phi}{M} \frac{\partial}{\partial m'} [\frac{\rho_\phi^2}{\alpha M} \frac{\partial p_\phi}{\partial m'}]$$

(55). (a).

Both ζ and Ψ being defined as above [Equations (31) and (49), respectively], we obtain

$$\frac{\partial \Psi}{\partial \tau'} = -[\frac{1}{M^2} \frac{\partial \Psi}{\partial m'} - \frac{m'}{M} \frac{dM}{d\tau'}] \frac{\partial \Psi}{\partial m'} + \frac{\rho_\phi^3}{M^2 \zeta \alpha} \frac{\partial^2 \Psi}{\partial m'^2} \qquad (56). (a).$$

It will be obvious that the solution of Equation (40) or Equation (56) must be started with a system of finite extent or mass already present, and in some prescribed initial state. For reasons of convenience, the same will ordinarily be assumed in solving Equation (35) or Equation (51). In solving Equation (35) or Equation (40), it will be necessary to have available the inverse

of the function $\Psi(p_\phi)$, so that the coefficient of the second derivative may be calculated; likewise, in solving Equation (51) or Equation (56), the inverse of $\Psi(p_\phi)$ will be needed.

In an application of the theory represented by Equation (56) to a formation problem, the function $M(t)$ may not be accurately known in advance; for example, one might have only the information that the solid fraction in the suspension approaching the system is constant, and that the fluid outflow rate $-U_\lambda(0,t)$ is known. An iterative procedure may be followed, beginning with the construction of a first approximation $M^{(1)}(t)$ which increases at the rate

$$\rho_\phi (1 - \varepsilon_s) U_\lambda (0,t) / \varepsilon_s \qquad (57),$$

where $1 - \varepsilon_s$ is the solid fraction in the suspension.

IMPROVED APPROXIMATION

In conclusion, we may indicate the way in which one of the inertial effects could be included. Returning to Equation (20), let us suppose that the coefficients α and β are both different from zero (they are, in fact, non-negative), and, for purposes of illustration, let the relative superficial velocity be negative. After solving Equation (20) for the relative superficial velocity, we find

$$U_\lambda (x,t) = \varepsilon U_\lambda (0,t) - \frac{1 - \varepsilon}{2\beta} \alpha \left[\left(1 + \frac{4\beta}{\alpha^2} \frac{\partial p_\lambda}{\partial x} \right)^{1/2} - 1 \right] \qquad (58)\text{(a)},$$

with the aid of Equation (26), and the result shown in Equation (58) may now be substituted in Equation (3).

When this inertial effect is appreciable, but not large, the radical in Equation (58) may be replaced by a series expansion. Thus

$$U_{\lambda}(x,t) = \varepsilon U_{\lambda}(0,t) - \frac{1-\varepsilon}{\alpha}\frac{\partial p_{\lambda}}{\partial x} + \frac{\beta(1-\varepsilon)}{\alpha^3}\left[\frac{\partial p_{\lambda}}{\partial x}\right]^2 + \ldots \qquad (59)\ (a),$$

which may be compared with Equation (27).

PART II. RETENTION OF SMALL PARTICLES IN DEFORMABLE POROUS SOLIDS

DEFINITIONS

As in the first part of this report, the system to be considered consists of a porous solid and a fluid which flows through it. We will suppose that the fluid is incompressible, in the sense that its true density does not change, and that the same may be said of the solid; however, the porous solid may be deformable. The fluid, as it approaches the region occupied by the system, may carry in suspension particles identical with those which compose the porous solid; it will be assumed that such particles are invariably intercepted and added to the solid phase of the system at its upstream boundary. The total mass of the porous solid will then increase with time. For brevity, we will refer to this process as <u>filtration</u>, interpreting the term in a narrow and specific sense.

We now consider, however, that the approaching fluid may also carry in suspension particles of a second kind, which are assumed to be <u>small</u> in each of the following respects: (a) the volume occupied by these particles, per unit total volume, is negligible even at the greatest concentration of such particles which is to be encountered; (b) the particles are to be of negligible dimensions compared to those of the smallest passages in the porous solid, through which any significant part of the flow takes place; (c) the particles of the second kind are to be very small compared to those which make up the porous solid. It is supposed that the small particles originally in suspension may become attached to exposed surfaces of the porous solid as the fluid flows through it, a process which will be referred to as <u>retention</u> (a term which is thus to be used, in the context of this discussion, with a restricted meaning).

It will be assumed that the presence of the particles of the second kind does not modify the flow of the fluid at any point; thus, we shall assume the applicability (and availability) of solutions of the corresponding flow problem in which the particles of the second kind are not present, as set forth in the preceding sections of this report.

The retention of small particles under these circumstances will be described with the aid of new variables, which represent the fluxes and concentrations of small particles which are, respectively, free, and bound. These terms correspond to the necessary distinction between small particles in suspension in the fluid (free) and those which have become attached to the porous solid (bound).

We shall present no account of the mechanisms, through which the attachment of the small particles to the exposed surfaces of the porous solid is effected, nor of the various factors which may influence the result. The disappearance of small particles from the fluid, by attachment, will be described phenomenologically, by means of a "capture cross section" for this process. That such a description is appropriate is therefore one of the hypotheses of this discussion.

The general conclusions to be presented consist of an additional continuity condition, relating to the conservation of small particles, and equations from which the free- and bound-particle concentrations may be determined. No assumptions are made as to the details of the structure of the porous solid, excepting those which are implicit in the use of local averages to describe the system. Attention is given to the special case of a porous solid consisting of approximately identical cylindrical fibers lying essentially in random directions in planes transverse to the direction of flow.

The results are illustrated by application to the problem of retention during slow filtration, for which an explicit solution is obtained.

THE RELATION BETWEEN FLUXES AND CONCENTRATIONS OF SMALL PARTICLES

To describe the transport of small particles in the system, we may define both the free-particle flux Q_λ and the bound-particle flux Q_ϕ in terms of the number of small particles carried across a plane perpendicular to the x axis, in the direction of increasing x, per unit area and per unit time. Only the mechanism of transport is different in the two cases: the free-particle flux results from the motion of the fluid, which carries unattached small particles with it; the bound-particle flux results from motion of the porous solid.

We also define the free-particle concentration C_λ as the number of small particles in suspension in the fluid, per unit volume of the fluid, and the bound-particle concentration C_ϕ as the number of attached small particles, per unit volume of the porous solid. It should be emphasized that these concentrations, as well as the fluxes defined above, are to be understood as local averages.

It is necessary to establish a connection between each flux and the corresponding concentration, by way of other known quantities. We may, for example, define functions Γ_λ and Γ_ϕ such that the relations

$$Q_\lambda = \Gamma_\lambda U_\lambda C_\lambda \tag{60}$$

and

$$Q_\phi = \Gamma_\phi U_\phi C_\phi \tag{61}$$

hold at each value of x and t. While it may be expected that these dimensionless functions will not depend explicitly on the time, and that they will not differ

from unity by orders of magnitude, it is by no means obvious that a satisfactory approximation to typical experimental conditions will result from setting $\Gamma_\lambda = 1$ and $\Gamma_\phi = 1$. Since we intend, in later sections, to develop in detail the consequences of assuming these values (thereby effecting an appreciable simplification of the analysis), the nature of the approximation must be considered here.

Given a sufficiently detailed model, it would be possible, in principle, to find the microscopic velocities of the fluid and the porous solid, together with the microscopic small particle concentrations, at each point in the system, at a given time. Let us consider the intersection of the system with a transverse plane at x; at points in this plane, let u_λ be the x component of the microscopic fluid velocity, and let c_λ be the microscopic concentration of free small particles. The quantities $\delta u_\lambda = u_\lambda - v_\lambda$ and $\delta c_\lambda = c_\lambda - C_\lambda$ are thus the departures from the local average values. Choose a region of area S in the transverse plane at x; let S' be the part of S which consists of cross sections of flow passages, and let S'' be the part of S which consists of cross sections of particles of the porous solid, together with cross sections of interstices which are inaccessible to flow. Thus, $S = S' + S''$; moreover, if S is of sufficient extent, and the arrangement of the particles of the porous solid is irregular, we have $S'/S = \varepsilon$ and $S''/S = 1 - \varepsilon$. The local average represented by Q_λ is given by

$$Q_\lambda = (1/S) \iint_{S'} (\delta u_\lambda + v_\lambda)(\delta c_\lambda + C_\lambda)\,dA \qquad (62),$$

or

$$Q_\lambda = (1/S) \iint_{S'} \delta u_\lambda \, \delta c_\lambda \, dA + (S'/S) v_\lambda \, C_\lambda \qquad (63).$$

In Equation (63), the last quantity on the right can be recognized as the product $U_\lambda C_\lambda$, and we have the conclusion that a lack of correlation between departures from local average values, in the variables u_λ and c_λ, implies that $\Gamma_\lambda = 1$. The condition is both sufficient and necessary. In particular, this value will result if δc_λ is zero or negligible throughout.

It is believed that porous solids consisting of small granules or fibers, in which the flow passages are highly irregular and frequently interconnected, will exhibit such effective remixing that we may take $\Gamma_\lambda = 1$ for the latter reason. This, however, is to be regarded as a physical hypothesis, subject to verification.

It should be said, also, that counter examples can be constructed which lead to other values. Thus, a model filter having long, narrow flow passages, without interconnections, assuming diffusion-limited retention and laminar flow, can be shown to involve a quite appreciable correlation; the value of Γ_λ is nearly 3.

Similarly, we may display the meaning of the function Γ_ϕ in Equation (61): We define u_ϕ as the x component of the microscopic velocity of the porous solid, and c_ϕ as the microscopic concentration of bound small particles; the departures from local average values are now $\delta u_\phi = u_\phi - v_\phi$ and $\delta c_\phi = c_\phi - C_\phi$. The local average represented by Q_ϕ is given by

$$Q_\phi = (1/S) \iint_{S''} (\delta u_\phi + v_\phi)(\delta c_\phi + C_\phi) dA \qquad (64),$$

or

$$Q_\phi = (1/S) \iint_{S''} \delta u_\phi \, \delta c_\phi dA + (S''/S) v_\phi C_\phi \qquad (65).$$

Since the last quantity on the right in Equation (65) is $U_\phi C_\phi$, we conclude that a lack of correlation between departures from local average values, in the variables u_ϕ and c_ϕ, implies that $\Gamma_\phi = 1$. The condition is, again, both sufficient and necessary. This value will obtain, in particular, if δu_ϕ is zero or negligible throughout.

It is for this reason that it is believed permissible to take $\Gamma_\phi = 1$ when the porous solid consists of small granules or fibers, in which it must be expected that displacements at nearby points are, in general, nearly equal.

CONSERVATION OF SMALL PARTICLES

The continuity condition relating to the small particles must take into account the possible transformation from the free to the bound state (or vice versa) during flow. It must therefore contain both fluxes and both concentrations. The derivation proceeds in the usual way, by consideration of inflow, outflow, and accumulation in a region having the form of a shallow cylinder, bounded above and below by transverse planes. The result is

$$\frac{\partial}{\partial x}\left[Q_\lambda + Q_\phi\right] = -\frac{\partial}{\partial t}\left[\varepsilon C_\lambda + (1-\varepsilon)C_\phi\right] \qquad (66).$$

When it is assumed that $\Gamma_\lambda = 1$ and $\Gamma_\phi = 1$, Equation (66) becomes

$$\frac{\partial}{\partial x}\left[U_\lambda C_\lambda + U_\phi C_\phi\right] = -\frac{\partial}{\partial t}\left[\varepsilon C_\lambda + (1-\varepsilon)C_\phi\right] \qquad (67).$$

Equation (67) may be rewritten, with the aid of Equations (3) and (4), as

$$U_\lambda \frac{\partial C_\lambda}{\partial x} + U_\phi \frac{\partial C_\phi}{\partial x} = -\varepsilon \frac{\partial C_\lambda}{\partial t} - (1-\varepsilon)\frac{\partial C_\phi}{\partial t} \qquad (68).$$

The following alternative arrangement of Equation (68) emphasizes the analogy to expressions for total derivatives evaluated at moving points:

$$\varepsilon \frac{\partial C_\lambda}{\partial t} + U_\lambda \frac{\partial C_\lambda}{\partial x} = -\left[(1-\varepsilon)\frac{\partial C_\phi}{\partial t} + U_\phi \frac{\partial C_\phi}{\partial x}\right] \qquad (69).$$

SMALL PARTICLE REMOVAL BY ATTACHMENT

The assumptions to be made, and the notation, may most easily be explained if we consider, first, a system in which the porous solid is stationary and unconsolidated. Let a considerable number of the particles of the porous solid, located in or near a given transverse plane, be chosen at random, and labeled with an integral index i. Let the flux of small particles at the transverse plane, resulting from fluid motion, be Q_λ. We also require the projected area A_i (the projection being taken on the transverse plane) and the volume J_i of particle i, together with the number of small particles attached to it, denoted by P_i. The quantities involved being functions, in general, of x and t, where x is the co-ordinate of the transverse plane, we shall assume that the process of attachment is correctly described by the equation

$$\frac{d}{dt} \sum_i P_i = \left| Q_\lambda \right| E \sum_i A_i \qquad (70),$$

which also serves as the definition of the efficiency, E. Since the total projected area $\sum_i A_i$ cannot be negative, and the absolute value of the small particle flux Q_λ has been taken, the right-hand side of Equation (70) will have the same sign as E, which must therefore be positive if the number of attached small particles is increasing. A dependence of the result on the direction of the small particle flux, if present, is to be taken into account by a change in the value of E.

In the present notation, the definition of C_ϕ is

$$\Sigma_i P_i = C_\phi \Sigma_i J_i \qquad (71),$$

from which

$$\frac{dC_\phi}{dt} = \left| Q_\lambda \right| E \frac{\Sigma_i A_i}{\Sigma_i J_i} = \left| Q_\lambda \right| E \frac{A}{J} \qquad (72),$$

where A is the average projected area of the particles of the porous solid, taken over the selected region, and J is the corresponding average volume.

It is agreed that the description of the process, represented by Equation (70) and its consequences, will be profitable only if it affords a separation of effects; thus, while Equation (70) is formally correct in all cases in which the attachment of small particles depends only on the variables explicitly shown, we hope to find, in practice, that E, thus defined, will be substantially independent of the magnitude of Q_λ. It is obvious for physical reasons that E will not depend explicitly on the total projected area.

As examples of the values of the ratio A/J, we find that a porous solid consisting of particles in the form of circular cylinders of the same diameter d_{cyl}, lying essentially in transverse planes, gives

$$\frac{A}{J} = \frac{4}{\pi d_{cyl}} \qquad (73),$$

while a porous solid consisting of spherical particles of the same diameter d_{sph}, gives

$$\frac{A}{J} = \frac{3}{2d_{sph}} \qquad (74).$$

It is convenient to define an attenuation coefficient K, as

$$K = (1 - \varepsilon)E \frac{A}{J} \qquad (75),$$

so that Equation (72) becomes

$$\frac{dC_\phi}{dt} = \left| Q_\lambda \right| \frac{K}{1 - \varepsilon} \qquad (76).$$

Equation (76) also serves as a <u>definition</u> of K in consolidated systems, where an appreciable part of the surface of the porous solid is not exposed to fluid flow and therefore cannot contribute to the collection of free particles.

We have now to consider the appropriate generalization of Equation (76), when the particles of the porous solid may be in motion. For an observer moving with the particles of the porous solid, at a given level, the result must be

$$\frac{dC_\phi}{dt} = \left| Q_\sigma \right| \frac{K}{1 - \varepsilon} \qquad (77),$$

where Q_σ is the <u>relative</u> free particle flux. This is related to the value of Q_λ, measured at the same location and time in the fixed co-ordinate system by

$$Q_\sigma = Q_\lambda - v_\phi \varepsilon C_\lambda = Q_\lambda - \frac{\varepsilon}{1 - \varepsilon} U_\phi C_\lambda \qquad (78).$$

We thus have

$$\frac{dC_\phi}{dt} = \left| Q_\lambda - \frac{\varepsilon}{1 - \varepsilon} U_\phi C_\lambda \right| \frac{K}{1 - \varepsilon} \qquad (79).$$

To secure the desired result, it is only necessary to convert the left-hand side of Equation (79) to quantities referring to the fixed co-ordinate system. We obtain

$$\frac{\partial C_\phi}{\partial t} + v_\phi \frac{\partial C_\phi}{\partial x} = \left| Q_\lambda - \frac{\varepsilon}{1-\varepsilon} U_\phi C_\lambda \right| \frac{K}{1-\varepsilon} \qquad (80),$$

or

$$(1-\varepsilon) \frac{\partial C_\phi}{\partial t} + U_\phi \frac{\partial C_\phi}{\partial x} = \left| Q_\lambda - \frac{\varepsilon}{1-\varepsilon} U_\phi C_\lambda \right| K \qquad (81).$$

Let the further assumption now be introduced, that $\Gamma_\lambda = 1$. Equation (81) becomes

$$(1-\varepsilon) \frac{\partial C_\phi}{\partial t} + U_\phi \frac{\partial C_\phi}{\partial x} = \left| U_\lambda - \frac{\varepsilon}{1-\varepsilon} U_\phi \right| C_\lambda \; K \qquad (82),$$

in which the first factor on the right-hand side can be recognized as $\left| U_\sigma \right|$, where U_σ is the relative superficial velocity.

When it is also assumed that $\Gamma_\phi = 1$, we may apply Equation (68) so as to obtain a relation involving C_λ alone:

$$\varepsilon \frac{\partial C_\lambda}{\partial t} + U_\lambda \frac{\partial C_\lambda}{\partial x} = -\left| U_\lambda - \frac{\varepsilon}{1-\varepsilon} U_\phi \right| C_\lambda \; K \qquad (83).$$

The procedure to be followed in obtaining calculated values of C_λ and C_ϕ, as functions of x and t, thus consists of substitution of the solution of the flow problem in Equation (83), which is then solved with appropriate initial and boundary conditions. The result is then inserted in Equation (82), which is solved in turn.

As in the flow problem, numerical methods of solution will be required in nearly all cases. With suitable co-ordination, the three numerical integrations can be carried forward, in a sense, simultaneously; it is not necessary to have

available a complete schedule of $U_\lambda(x,t)$ and $U_\phi(x,t)$ before beginning the solution of Equation (83), etc.

ALTERNATIVE CO-ORDINATE SYSTEM

In this section, formulas corresponding to Equations (82) and (83) are derived, to describe the retention process (under the same hypotheses) in other co-ordinate systems, which are sometimes convenient in applications. The transformation process being similar in each instance, the results are stated without lengthy explanation. To avoid an irrelevant complication, the previous definitions of U_λ, U_ϕ, and U_σ are retained; these quantities have therefore the values measured in the stationary x, t system, and appropriate substitutions are made for the variables x and t.

In a co-ordinate system in which the independent variables are X, the distance below the upstream boundary of the porous solid (of which the total thickness is L), and T, the time, we have

$$X = L - x ; \qquad T = t \tag{84},$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} = -\frac{\partial}{\partial X} \tag{85},$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T} = \frac{dL}{dt}\frac{\partial}{\partial X} + \frac{\partial}{\partial T} \tag{86}.$$

With the abbreviation

$$V = \frac{dL}{dt} \tag{87},$$

(this being the velocity of the upstream boundary with respect to the downstream boundary), we obtain, as the equivalent of Equation (82),

$$(1 - \varepsilon)\frac{\partial C_\phi}{\partial T} - [U_\phi - (1 - \varepsilon)V]\frac{\partial C_\phi}{\partial X} = |U_\sigma| C_\lambda K \qquad (88),$$

and, as the equivalent of Equation (83),

$$\varepsilon \frac{\partial C_\lambda}{\partial T} - [U_\lambda - \varepsilon V]\frac{\partial C_\lambda}{\partial X} = -|U_\sigma| C_\lambda K \qquad (89).$$

Another useful description employs the "relative distance" system of co-ordinates, defined by

$$x' = \frac{x}{L}; \qquad t' = t \qquad (90).$$

We now have

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \qquad (91),$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{x'}{L}\frac{dL}{dt'}\frac{\partial}{\partial x'} \qquad (92).$$

The result corresponding to Equation (82) is

$$(1 - \varepsilon)\frac{\partial C_\phi}{\partial t'} + \left[\frac{U_\phi}{L} - (1 - \varepsilon)\frac{x'}{L}\frac{dL}{dt'}\right]\frac{\partial C_\phi}{\partial x'} = |U_\sigma| C_\lambda K \qquad (93),$$

and that corresponding to Equation (83) is

$$\varepsilon \frac{\partial C_\lambda}{\partial t'} + \left[\frac{U_\lambda}{L} - \varepsilon \frac{x'}{L} \frac{dL}{dt'} \right] \frac{\partial C_\lambda}{\partial x'} = - \left| U_\sigma \right| C_\lambda K \qquad (94).$$

We may next consider a system of co-ordinates in which the distance x is replaced by a "cumulative mass" m, which is the mass of the porous solid, per unit area of system cross section, included between transverse planes at x and at the origin. The co-ordinates are

$$m = \int_o^x \rho_\Phi (1 - \varepsilon) dx \; ; \qquad \tau = t \qquad (95).$$

We obtain

$$\frac{\partial}{\partial x} = \rho_\Phi (1 - \varepsilon) \frac{\partial}{\partial m} = \rho_\phi \frac{\partial}{\partial m} \qquad (96),$$

and

$$\frac{\partial}{\partial t} = \frac{\partial m}{\partial t} \frac{\partial}{\partial m} + \frac{\partial}{\partial \tau} \qquad (97),$$

where

$$\frac{\partial m}{\partial t} = \rho_\Phi [U_\lambda (x,t) - U_\lambda (0,t)] \qquad (98),$$

as was shown previously, in Equation (44). Equation (82) now becomes

$$(1 - \varepsilon) \frac{\partial C_\phi}{\partial \tau} + \rho_\phi [U_\lambda (x,t) - U_\lambda (0,t) + U_\phi] \frac{\partial C_\phi}{\partial m} = \left| U_\sigma \right| C_\lambda K \qquad (99),$$

or, on introducing Equation (7),

$$(1 - \varepsilon) \frac{\partial C_\phi}{\partial \tau} = \left| U_\sigma \right| C_\lambda K \qquad (100)\ (a).$$

Equation (83) becomes

$$\varepsilon \frac{\partial C_\lambda}{\partial \tau} + \rho_\Phi \left[U_\lambda (x,t) - \varepsilon U_\lambda (0,t) \right] \frac{\partial C_\lambda}{\partial m} = - \left| U_\sigma \right| C_\lambda K \qquad (101),$$

which can be somewhat shortened by use of Equation (26), giving

$$\varepsilon \frac{\partial C_\lambda}{\partial \tau} + \rho_\phi U_\sigma \frac{\partial C_\lambda}{\partial m} = - \left| U_\sigma \right| C_\lambda K \qquad (102)\ (a).$$

The simplicity of Equation (100) and its resemblance to Equation (77) result from the choice of a proper co-ordinate system, in that a fixed value of m corresponds to a point which moves with the nearby particles of the porous solid.

A co-ordinate system analogous to the X, T system results from the definitions

$$G = M - m ; \qquad T = \tau \qquad (103).$$

Here G is the cumulative mass of the porous solid, per unit area of system cross section, included between transverse planes at x and at the upper boundary. We find

$$\frac{\partial}{\partial m} = - \frac{\partial}{\partial G} \qquad (104),$$

and

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T} + \frac{dM}{dT} \frac{\partial}{\partial G} \qquad (105).$$

The equivalent of Equation (100) is

$$(1 - \varepsilon) \frac{\partial C_\phi}{\partial T} + (1 - \varepsilon) \frac{dM}{dT} \frac{\partial C_\phi}{\partial G} = \left| U_\sigma \right| C_\lambda K \qquad (106),$$

and the equivalent of Equation (101) is

$$\varepsilon \frac{\partial C_\lambda}{\partial T} + \left[\varepsilon \frac{dM}{dT} - \rho_\Phi \left\{ U_\lambda (x,t) - \varepsilon U_\lambda (0,t) \right\} \right] \frac{\partial C_\lambda}{\partial G} = - \left| U_\sigma \right| C_\lambda K \qquad (107).$$

From Equation (102) we obtain

$$\varepsilon \frac{\partial C_\lambda}{\partial T} - \left[\rho_\phi U_\sigma - \varepsilon \frac{dM}{dT} \right] \frac{\partial C_\lambda}{\partial G} = - \left| U_\sigma \right| C_\lambda K \qquad (108)\ (a).$$

Lastly, in a "relative cumulative mass" system, defined by

$$m' = \frac{m}{M}, \qquad \tau' = \tau \qquad (109),$$

we have

$$\frac{\partial}{\partial m} = \frac{1}{M} \frac{\partial}{\partial m'} \qquad (110),$$

and

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau'} - \frac{m'}{M} \frac{dM}{d\tau'} \frac{\partial}{\partial m'} \qquad (111).$$

The result corresponding to Equation (100) is

$$(1 - \varepsilon) \frac{\partial C_\phi}{\partial \tau'} - (1 - \varepsilon) \frac{m'}{M} \frac{dM}{d\tau'} \frac{\partial C_\phi}{\partial m'} = \left| U_\sigma \right| C_\lambda K \qquad (112),$$

and that corresponding to Equation (101) is

$$\varepsilon \frac{\partial C_\lambda}{\partial \tau'} + \left[\rho_\Phi \left\{ U_\lambda (x,t) - \varepsilon U_\lambda (0,t) \right\} - \varepsilon \frac{m'}{M} \frac{dM}{d\tau'} \right] \frac{\partial C_\lambda}{\partial m'} = - \left| U_\sigma \right| C_\lambda K \qquad (113).$$

From Equation (102), there follows

$$\varepsilon \frac{\partial C_\lambda}{\partial \tau'} + \left[\rho_\phi U_\sigma \frac{1}{M} - \varepsilon \frac{m'}{M} \frac{dM}{d\tau'} \right] \frac{\partial C_\lambda}{\partial m'} = - \left| U_\sigma \right| C_\lambda K \qquad (114)\ (a).$$

THE SPECIAL CASE OF SLOW FILTRATION

The phrase "slow filtration" will be defined as implying that the flow variables (pressure, porosity, internal superficial velocities) do not, at any moment or position, differ appreciably from those which would obtain in a system of similar contents, having a fixed total mass of porous solid per unit area of system cross section (equal to the instantaneous value of the corresponding quantity in the system of interest). Constant and uniform flow conditions are assumed in the comparison system, in which the outflow rate is taken to be equal, likewise, to the instantaneous value in the actual system. Thus, in "slow filtration" the system develops, in time, by passing through a succession of equilibrium states. As particular consequences of the definition, we conclude that $\left| U_\phi \right|$ must be small compared to $\left| U_\lambda \right|$ throughout the system, and that both U_λ and U_ϕ, regarded as functions of x, must be nearly constant.

Assuming a constant outflow rate, let us consider a system in which C_λ and C_ϕ, as measured in the suspension approaching the system, are known and constant. The values will be denoted by $(C_\lambda)_s$ and $(C_\phi)_s$, respectively. From the point of view of an observer moving with the upper boundary, who expresses the

result in the X, T or in the G, T co-ordinate system, it will appear that the values of C_λ do not depend on the time; at points within the system C_λ depends only on location with respect to the upper boundary. The same will obviously be true of the values of the pressure and porosity.

The G, T co-ordinate system turns out to be the most convenient. If, in Equation (108), it is assumed that C_λ depends only on G, we have

$$\left[\rho_\phi U_\sigma + \varepsilon \frac{dM}{dT}\right] \frac{\partial C_\lambda}{\partial G} = \left| U_\sigma \right| C_\lambda K \qquad (115).$$

If the second term in the bracket is negligible compared to the first, and we observe that U_σ cannot differ significantly from U_λ, it follows that

$$\frac{\partial C_\lambda}{\partial G} = -\frac{K}{\rho_\phi} C_\lambda = -\frac{K}{\rho_\Phi (1 - \varepsilon)} C_\lambda \qquad (116),$$

in which we have replaced $\left|U_\sigma\right|$ by $-U_\lambda$ (since the flow is supposed to take place in the negative x direction). Inserting the value of K, as given by Equation (75), this becomes

$$\frac{\partial C_\lambda}{\partial G} = -\frac{E}{\rho_\Phi} \frac{A}{J} C_\lambda \qquad (117).$$

If the multiplier of C_λ is a constant, this expression may be integrated in the form

$$C_\lambda = (C_\lambda)_s \exp\left[-\frac{E}{\rho_\Phi} \frac{A}{J} G\right] \qquad (118).$$

The corresponding values of C_ϕ are easily obtained by returning to the m, τ co-ordinate system. From Equation (100),

$$\frac{\partial C_\phi}{\partial \tau} = - U_\lambda C_\lambda \frac{K}{1 - \varepsilon} = - U_\lambda C_\lambda E \frac{A}{J} \qquad (119),$$

in which we may substitute the value of C_λ, from Equation (118), with the argument $G = M - m$:

$$\frac{\partial C_\phi}{\partial \tau} = -U_\lambda E \frac{A}{J} (C_\lambda)_s \exp[- \frac{E}{\rho_\Phi} \frac{A}{J} (M - m)] \qquad (120).$$

This result may now be integrated if the dependence of the parameter M on the time is known. Let us assume that $dM/d\tau$, which is the same as dM/dT, is a constant, and that the origin of the time scale is chosen so that $M = (dM/d\tau)\tau$. We obtain

$$C_\phi = - \frac{U_\lambda}{\frac{dM}{dT}} \rho_\Phi (C_\lambda)_s \left\{ 1 - \exp[- \frac{E}{\rho_\Phi} \frac{A}{J} (M - m)] \right\} + (C_\phi)_s$$

(121).

In the G, T co-ordinate system, it is seen that the result depends on G alone. Moreover, this feature is a consequence of the assumed constancy of dM/dT. The same result could have been obtained from Equation (106), by <u>assuming</u> that C_ϕ depends on G alone; this equation can then be true only if dM/dT is constant. The added assumption, however, makes the argument less convincing. It should be noted that cases of slow filtration in which dM/dt is not (quite) constant, but the other assumptions preceding Equation (120) are justified, can be accounted for by integration of Equation (120) with the appropriate form of $M(\tau)$.

Perhaps the chief point of interest in the explicit solution, Equation (121), is that it does <u>not</u> depend on any assumptions regarding deformability.

THE DISTINCTION BETWEEN SLOW AND RAPID FILTRATION

A somewhat more quantitative indication may be derived, showing whether a given example of filtration is "slow," in the sense of the previous section. In an X, T co-ordinate system, we have

$$M = \int_o^L \rho_{\bar{\Phi}} [1 - \varepsilon(X,T)]dX \qquad (122),$$

so that

$$\frac{dM}{dT} = \rho_{\bar{\Phi}} [1 - \varepsilon(L,T)]V - \rho_{\bar{\Phi}} \int_o^L \frac{\partial \varepsilon}{\partial T} dX \qquad (123),$$

where $V = dL/dT$. This may be restated in the x,t co-ordinate system, with the aid of Equations (85) and (86), as

$$\frac{dM}{dt} = \rho_{\bar{\Phi}} [1 - \varepsilon(0,t)]V - \rho_{\bar{\Phi}} \int_o^L \left[\frac{\partial \varepsilon}{\partial t} + V \frac{\partial \varepsilon}{\partial x}\right] dx \qquad (124).$$

Introducing Equation (3), and integrating,

$$\frac{dM}{dt} = \rho_{\bar{\Phi}} [1 - \varepsilon(0,t)]V - \rho_{\bar{\Phi}} V[\varepsilon(L,t) - \varepsilon(0,t)] + \rho_{\bar{\Phi}} [U_\lambda(L,t) - U_\lambda(0,t)] \qquad (125).$$

The first term in this expression is the value given for dM/dt in the "slow filtration" model, for the effect is that of an increase in mat thickness at the rate V, as if material were being added to the porous solid at the porosity prevailing at the lower boundary.

Thus, as a necessary condition, the contribution from the other terms must be small, compared to the first, throughout the time interval of interest. The condition is

$$\left| 1 - \frac{U_{\lambda}(L,t) - U_{\lambda}(0,t)}{V[1 - \varepsilon(0,t)]} \right| << 1 \qquad (126),$$

which represents a limitation on the allowable variation in the internal superficial velocity at the upper and lower boundaries.

NOMENCLATURE

In connection with each symbol, a reference is given to the equation or page where the quantity is defined or first used. Symbols listed without comment are empirical constants.

Symbol	Equation	Page	Description
a	22		
A		28	Average projected area of particles of porous solid
A_i		27	Projected area of particle i of porous solid
b	22		
b'	23		
c_o	24		
c_1	24		
c_λ		24	Microscopic concentration of free small particles
c_ϕ		25	Microscopic concentration of bound small particles
C_λ		23	Number of free particles per unit volume of fluid
$(C_\lambda)_s$		36	Value of C_λ in approaching suspension, immediately above system
C_ϕ		23	Number of bound particles per unit volume of porous solid
$(C_\phi)_s$		36	Value of C_ϕ in approaching suspension, immediately above system
d_{cyl}		28	Diameter of cylindrical porous solid particle
d_{sph}		28	Diameter of spherical porous solid particle
E	70		Efficiency
F_λ	11		Force required to accelerate fluid, per unit volume
F_ϕ	12		Force required to accelerate solid, per unit volume
G	103		$G = M - m$

Symbol	Equation	Page	Description
J		28	Average volume of particles of porous solid
J_i		27	Volume of particle i of porous solid
K	75		Attenuation coefficient
L		2	Thickness of system
m	41		Cumulative mass of porous solid, per unit area
m'		18	$m' = m/M$
M	52		Total mass of porous solid, per unit area
N	24		
p_λ		3	Pressure in fluid phase
$p_\lambda(L,t)$		5	Upstream fluid pressure
$p_\lambda(0,t)$		5	Downstream fluid pressure
p_ϕ		3	Pressure in solid phase
$p_\phi(L,t)$		5	Pressure on upper boundary of solid phase
$p_\phi(0,t)$		5	Pressure on lower boundary of solid phase
P_i		27	Number of small particles attached to particle i of porous solid
Q_λ		23	Free particle flux
Q_ϕ		23	Bound particle flux
Q_σ	78		Relative free particle flux
r		8	Local flow resistance
S		24	Area in transverse plane
S'		24	Area in transverse plane
S''		24	Area in transverse plane
S_V	22		Specific surface of solid phase
t		2	Time
t'		15	Time

Symbol	Equation	Page	Description
T	84		Time
u_λ		24	Component of microscopic fluid velocity, in x direction
u_ϕ		25	Component of microscopic solid velocity, in x direction
U_λ		3	Internal superficial velocity of fluid
$-U_\lambda\,(L,t)$		5	Fluid inflow rate per unit area
$-U_\lambda\,(0,t)$		5	Fluid outflow rate per unit area
U_ϕ		3	Internal superficial velocity of solid
U_σ	17		Relative superficial velocity of fluid
v_λ		3	Averaged microscopic fluid velocity
v_ϕ		3	Averaged microscopic solid velocity
V	87		Velocity of upper boundary of system
υ	22		Specific volume of solid phase
w		11	Arbitrary velocity
w_1		10	Arbitrary velocity
w_2		10	Arbitrary velocity
x		2	Distance
x'		15	$x' = x/L$
x_I	41		Variable of integration
X	84		$X = L - x$
α	15		Function of porosity and other system parameters
β	19		Function of porosity and other system parameters
Γ_λ	60		Dimensionless function
Γ_ϕ	61		Dimensionless function
δc_λ		24	$\delta c_\lambda = c_\lambda - C_\lambda$
δc_ϕ		25	$\delta c_\phi = c_\phi - C_\phi$

Symbol	Equation	Page	Description
δu_λ		24	$\delta u_\lambda = u_\lambda - v_\lambda$
δu_ϕ		25	$\delta u_\phi = u_\phi - v_\phi$
ε		3	Porosity, or fluid fraction
ε_s	57		Fluid fraction in approaching suspension
ζ	31		An abbreviation
μ	22		Fluid viscosity
Π		7	Force exerted on solid phase by fluid phase, per unit volume of system
ρ_λ		3	Apparent density of fluid
ρ_Λ		3	True density of fluid
ρ_ϕ		3	Apparent density of solid
ρ_Φ		3	True density of solid
τ	41		Time
τ'		18	Time
ψ	33		Modified dependent variable
Ψ	49		Modified dependent variable

A consistent set of units is assumed, such as the c.g.s. dynamical system.

Note that, as here defined, the density ρ_Φ refers to the mass within the exterior boundary of a particle of the solid phase, divided by the included volume; thus, when the problem involves the flow of a liquid through fibers which absorb the liquid, the <u>wet</u> mass and the <u>wet</u> volume of the fiber are called for. Similarly, m is the cumulative <u>wet</u> mass of solid per unit area. On the other hand, $\mathcal{V}$ is defined, by custom, as the <u>wet</u> volume of fiber divided by the <u>dry</u> mass.

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Part II of this report arose from attempts to interpret experimental studies of retention in deformable filters under conditions of slow filtration. These studies, and their interpretation, have been described in another report (4) in this series, by S. T. Han. The writer is indebted to him for much discussion and advice.

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